Heat kernels for isotropic-like Markov generators on ultrametric spaces: a survey

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1 Introduction

A systematic study of *isotropic* Markov semigroups defined on *ultrametric* measure spaces has been done in:

- A. Bendikov, A. Grigoryan and C. Pittet, On a class of Markov semigroups on discrete ultrametric spaces, Potential Analysis 37 (2012), 125-169,
- A. Bendikov, A. Grigoryan, C. Pittet and W. Woess, Isotropic Markov semigroups on ultrametric spaces., Russian Math. Surveys 69:4, 589-680 (2014).

This study was motivated by the theme *Random walks on infinitely generated groups*, the classical topic which can be traced back to the pioneering works of Erdös, Spitzer, Kesten, Cartwright, Molchanov, Lawler and others.

The notion of isotropic Markov semigroup acting on a *discrete* ultrametric measure space is closely related to the concept of the *hierarchical lattice and hierarchical Laplacian* introduced in the celebrated Dyson's paper.

- F.J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet, Comm. Math. Phys. (1969), 12: 91-107.

Namely, given an isotropic Markov semigroup defined on ultrametric measure space (X, d, m), one shows, that its minus Markov generator L is a hierarchical Laplacian defined in terms of the hierarchical lattice (i.e. the tree of metric balls) on (X, d, m), and vice versa.

- S. A. Molchanov, Hierarchical random matrices and operators, Application to Anderson model, Proc. of 6th Lucacs Symposium (1996), 179-194,

- A. Bendikov and P. Krupski, On the spectrum of the hierarchical Laplacian., Potential Analysis 41 (2014), no. 4, 1247-1266.

According to the general theory any isotropic Markov semigroup $(e^{-tL})_{t>0}$ admits a continuous transition density $p(t, x, y)$ w.r.t. m. We call $p(t, x, y)$ *the heat kernel*. Modifying canonically the underlying ultrametric d, we denote this new ultrametric d_* and call it *the intrinsic ultrametric*, one shows that

$$Lf(x) = \int_X (f(x) - f(y))J(x, y)dm(y), \tag{1.1}$$

$$J(x, y) = \int_0^{1/d_*(x,y)} N(x, \tau)d\tau \tag{1.2}$$

and

$$p(t, x, y) = t \int_0^{1/d_*(x,y)} N(x, \tau) \exp(-t\tau)d\tau. \tag{1.3}$$

Here $N(x, \tau)$ is *the spectral function* and $J(x, y)$ is *the jump kernel* related to L (the functions d_*, N and J will be defined later).

Notice that the families of d-balls and d_*-balls coincide, whence these two ultrametrics generate the same topology and the same hierarchical lettice (i.e. the tree of metric balls), and in particular, the same class of hierarchical Laplacians.

The aim of this lecture is to present recent results on two-sided estimates for heat kernels which are associated with certain Markov generators of the form (1.1) *having jump kernels uniformly comparable to the jump kernels associated with hierarchical Laplacians.*[1]

In the course of study we apply recent results due to Z.-Q. Chen, A. Grigor'yan, E. Hu, T. Kumagai, J. Wang and others about heat kernels related to non-local Dirichlet forms on metric spaces.

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2 Hierarchical v.v. isotropic Laplacian

Hierarchical lattice Let (X, d) be a locally compact and separable ultrametric space. Recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{2.1}$$

One of the basic consequences of the ultrametric property is that

- each open ball is a closed set,
- each point x of a ball B can be regarded as its center,
- any two balls A and B either do not intersect or one is a subset of another, etc.

In what follows we assume that the ultrametric space (X, d) is *proper*, that is, each closed d-ball is a compact set.

Example of Molchanov Consider $X = \mathbb{R}^1$, the set of reals equipped with Lebesgues measure m. Let us fix an integer $p \geq 2$ and consider a family $\{\Upsilon_r : r \in \mathbb{Z}\}$ of partitions of X :

$$\Upsilon_r = \{(kp^r, (k+1)p^r] : k \in \mathbb{Z}\}.$$

We call r the rank of the partition Υ_r (resp., the rank of the interval $I \in \Upsilon_r$). Each interval of rank r is the union of p disjoint intervals of rank $(r-1)$, each point $x \in X$ belongs to a certain interval $I_r(x)$ of rank r, and

$$I_{r-1}(x) \subset I_r(x) \subset I_{r+1}(x) \ \text{ and } \ \{x\} = \cap_{r\in\mathbb{Z}} I_r(x).$$

The hierarchical distance $d(x, y)$ is defined as follows:

$$d(x, y) = p^{\mathfrak{n}(x,y)}, \ \text{ where } \mathfrak{n}(x, y) = \inf\{r : y \in I_r(x)\}.$$

Notice that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and for arbitrary $z \in X$,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\},$$

i.e. $d(x, y)$ is an ultrametric.

The set X equipped with the ultrametric $d(x, y)$ is complete, separable and proper ultrametric space. In the ultrametric space (X, d) the set of all non-singletone balls coincides with the set of all p-adic intervals.

Hierarchical Laplacian Let $\mathcal{B} \subset X$ be the set of all non-singletone balls and $\mathcal{B}(x) \subset \mathcal{B}$ the set of all balls centred at x. The set $\mathcal{B}$ is atmost countable whereas X by itself may well be uncountable, e.g. $X = \mathbb{R}^1$ as in the example above. Let $C : \mathcal{B} \to (0, \infty)$ be a function such that for all $B \in \mathcal{B}$,

$$\lambda(B) := \sum_{T \in \mathcal{B}:\ B \subseteq T} C(T) < \infty \tag{2.2}$$

and, for all non-isolated $x \in X$,

$$\sup\{\lambda(B) : B \in \mathcal{B}(x)\} = \infty. \tag{2.3}$$

Let $\mathcal{D}$ be the set of all locally constant functions having compact support. The set $\mathcal{D}$ belongs to Banach spaces $C_0(X)$ and $L^p = L^p(X, m)$, $1 \leq p < \infty$, and is a dence subset there. For each $f \in \mathcal{D}$ and $x \in X$ we define (pointwise) the *hierarchical* Laplacian L_C as follows,

$$L_C f(x) := \sum_{B \in \mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f dm \right). \tag{2.4}$$

The operator $(L_C, \mathcal{D})$ acts in L^2, is symmetric and admits a complete system of eigenfunctions f_B,

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}, \tag{2.5}$$

where $B \subset B'$ run over all nearest neighboring balls having positive measure. The eigenvalue corresponding to f_B is the number $\lambda(B')$ defined at (2.2),

$$L_C f_B(x) = \lambda(B') f_B(x).$$

Since all $f_B \in \mathcal{D}$ and the system $\{f_B\} \subset L^2$ is complete we conclude that $(L_C, \mathcal{D})$ is essentially self-adjoint operator, i.e. has a unique self-adjoint extension.

For $x, y \in X$ we denote $x \curlywedge y$ the minimal ball containing x and y. The intrinsic ultrametric $d_*(x, y)$ is defined as follows,

$$d_*(x,y) := \begin{cases} 0 & \text{when} \quad x = y \\ 1/\lambda(x \curlywedge y) & \text{when} \quad x \neq y \end{cases}. \tag{2.6}$$

Notice that the ultrametrics d and d_* generate the same set of balls and that

$$\lambda(B) = \frac{1}{\operatorname{diam}_*(B)}, \quad \text{for all balls } B.$$

In general setting some eigenvalues may well have finite multiplicity and some not. Indeed, attached to each ball B of d_*-diameter $1/\lambda$ there are the eigenvalue λ and the corresponding eigenspace $\mathcal{H}_B$. The eigenspace $\mathcal{H}_B$ is spanned by finitely many functions f_T where $T \subset B$ runs over the finite set of all nearest neighboring balls of B. Let $n(B)$ be the cardinality of this set, then

$$\dim \mathcal{H}_B = n(B) - 1.$$

For two different balls B and C the eigenspaces $\mathcal{H}_B$ and $\mathcal{H}_C$ are orthogonal. As the set of all eigenfunctions is complete we conclude that

$$L^2(X, m) = \bigoplus_{B \in \mathcal{B}} \mathcal{H}_B.$$

The spectral function $\tau \to N(x, \tau)$ is defined as the left-continuous step-function having jumps at the points $\lambda(B)$, where B runs over the set of all balls centred at x, and such that

$$N(x, \lambda(B)) = 1/m(B).$$

The volume function $V(x, r)$ is defined as the volume of a ball centred at x and having d_*-radius r. The following equation holds

$$V(x, r) = 1/N(x, 1/r). \tag{2.7}$$

The heat kernel $p(t, x, y)$ is a continuous off the diagonal function which can be represented in the form

$$p(t, x, y) = t \int_0^{1/d_*(x,y)} N(x, \tau) \exp(-t\tau) d\tau. \tag{2.8}$$

It follows that if the function $\tau \to N(x, \tau)$ is *doubling* (and only in this case!),

$$p(t, x, y) \asymp \frac{t}{t + d_*(x, y)} N\left(x, \frac{1}{t + d_*(x, y)}\right), \tag{2.9}$$

uniformly in $y \in X$ and $t > 0$.

In turn, equations (2.7) and (2.9) imply the following result

$$p(t, x, y) \asymp \min\left\{\frac{1}{V(x, t)}, \frac{t}{V(x, d_*(x, y)) d_*(x, y)}\right\} \tag{2.10}$$

uniformly in $y \in X$ and $t > 0$.

Example 2.1 *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. For any two nearest neighbouring balls $B \subset B'$ we set*

$$C(B) = \phi\left(\frac{1}{m(B)}\right) - \phi\left(\frac{1}{m(B')}\right).$$

Then the following properties hold:

(i) $\lambda(B) = \Phi\left(1/m(B)\right)$,

(ii) $d_*(x,y) = 1/\Phi\left(1/m(x \curlywedge y)\right)$,

(iii) $V(x,r) \leq 1/\Phi^{-1}(1/r)$. Moreover, $V(x,r) \asymp 1/\Phi^{-1}(1/r)$ whenever the function $r \to \Phi(r)$ is reverse doubling and $m(B') \asymp m(B)$ for all neighboring balls $B \subset B'$ centred at x. In this case $r \to V(x,r)$ is doubling whence by (2.10), uniformly in $y \in X$ and $t > 0$,

$$p(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{m(x \curlywedge y)}\Phi\left(\frac{1}{m(x \curlywedge y)}\right)\right\}.$$

Isotropic nature of the hierarchical Laplacian Given a hierarchical Laplacian L_C as defined at (2.4) let us introduce two functions:

$$J(B) = \sum_{T \in \mathcal{B}: B \subseteq T} \frac{C(T)}{m(T)} \quad \text{and} \quad J(x,y) = J(x \curlywedge y). \tag{2.11}$$

It is remarkable but easy to prove that in the introduced notation

$$L_C f(x) = \int_X \left(f(x) - f(y)\right) J(x,y) dm(y), \tag{2.12}$$

i.e. $(L_C, \mathcal{D})$ coincides with certain integral operator having isotropic kernel, we call this operator *isotropic Laplacian.* Spectral theory of such operators was studied in the paper

- S. V. Kozyrev, Wavlets and spectral analysis of ultrametric pseudo-differential operators. Mat. Sb. (2007), 198:1 97-116.

Recall that $C(T) = \lambda(T) - \lambda(T')$, whence applying the Abel transform in equation (2.11) we get

$$J(B) = \frac{\lambda(B)}{m(B)} - \sum_{T \in \mathcal{B}: B \subseteq T} \lambda(T')\left(\frac{1}{m(T)} - \frac{1}{m(T')}\right) \leq \frac{\lambda(B)}{m(B)},$$

or

$$J(x,y) \leq \frac{1}{V(x, d_*(x,y))d_*(x,y)} \text{ uniformly in } x, y. \tag{2.13}$$

Equation (2.10) implies that if $\tau \to N(x,\tau)$ is *doubling* then also for some constant $\Xi > 0$,

$$J(x,y) \geq \frac{\Xi}{V(x, d_*(x,y))d_*(x,y)} \text{ uniformly in } y. \tag{2.14}$$

The other way round, consider a function $J(B)$ satisfying the following three conditions:

(J1) $S \subset T \Longrightarrow J(S) > J(T)$ and $J(T) \to 0$ as $T \to \varpi$.

(J2) $\lambda(T) := \sum_{S \in \mathcal{B}\colon T \subseteq S} m(S)(J(S) - J(S')) < +\infty$ for all $T \in \mathcal{B}$.

(J3) $\sup\{\lambda(T) : T \in \mathcal{B}(x)\} = +\infty$ whenever x is not isolated.

Let us set $J(x,y) = J(x \curlywedge y)$ and define the *isotropic Laplacian*

$$L^J f(x) = \int_X (f(x) - f(y))\, J(x,y) dm(y). \tag{2.15}$$

The operator L^J coincides with certain hierarchical Laplacian L_C. Indeed, let us define a function $C : \mathcal{B} \to (0, \infty)$ as

$$C(B) = m(B)(J(B) - J(B'))$$

and consider the hierarchical Laplacian L_C as defined at (2.4), then

$$L^J f(x) = L_C f(x), \tag{2.16}$$

for all $f \in \mathcal{D}$ and $x \in X$.

Spectral multipliers Consider $X = \mathbb{Q}_p^l$, the Cartesian product of l copies of the ring of p-adic numbers $\mathbb{Q}_p$ equipped with its standard ultrametric $d(x,y) = |x-y|_p$. The couple $(\mathbb{Q}_p^l, \mathrm{d})$ becomes a proper ultrametric space if we set

$$|z|_p = \max\{|z_i|_p : i = 1, ..., l\} \text{ and } \mathrm{d}(x,y) = |x-y|_p \,.$$

Let m be the normed Haar measure on the Abelian group $\mathbb{Q}_p^l$, $L^2 = L^2(\mathbb{Q}_p^l, m)$ and $\mathcal{F} : f \to \widehat{f}$ the Fourier transform of function $f \in L^2$. It is known that $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a bijection.

Let $\Phi : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ be an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ we define as L^2-spectral multiplier, that is,

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(|\xi|_p)\widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p^l.$$

The operator $\Phi(\mathfrak{D})$ is a hierarchical Laplacian, whence it can be represented in the form

$$\Phi(\mathfrak{D})f(x) = \int\limits_{\mathbb{Q}_p} (f(x) - f(y))J_\Phi(x,y)dm(y), \quad f \in \mathcal{D}.$$

The eigenvalues $\lambda_\Phi(B)$ of the operator $\Phi(\mathfrak{D})$ and the intrinsic ultrametric $d_\Phi(x,y)$ are of the form

$$\lambda_\Phi(B) = \Phi\left(\frac{p}{\mathrm{diam}(B)}\right) \quad \text{and} \quad d_\Phi(x,y) = 1/\Phi\left(\frac{p}{|x-y|_p}\right). \tag{2.17}$$

The volume function $V_\Phi(x,r)$ satisfies the following equation

$$V_\Phi(s) \asymp (1/\Phi^{-1}(1/s))^l. \tag{2.18}$$

Let $p_\Phi(t,x,y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assume that $\Phi(\tau)$ is reverse doubling, then, by equation (2.10),

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\left(\Phi^{-1}\left(\frac{1}{t}\right)\right)^l, \left(\frac{p}{|x-y|_p}\right)^l \Phi\left(\frac{p}{|x-y|_p}\right)\right\} \tag{2.19}$$

and

$$J_\Phi(x,y) \asymp \left(\frac{p}{|x-y|_p}\right)^l \Phi\left(\frac{p}{|x-y|_p}\right) \tag{2.20}$$

uniformly in $t > 0$ and x, y.

As an example, $\Phi(\tau) = \tau^\alpha$, the operator $\mathfrak{D}^\alpha$ is a hierarchical Laplacian. Its heat kernel $p_\alpha(t,x,y)$ and its jump kernel $J_\alpha(x,y)$ satisfy

$$p_\alpha(t,x,y) \asymp \frac{t}{(t^{1/\alpha} + |x-y|_p)^{l+\alpha}}$$

and

$$J_\alpha(x,y) = \frac{p^\alpha - 1}{1 - p^{-l-\alpha}} \frac{1}{|x-y|_p^{l+\alpha}}.$$

3 Isotropic-like Markov generators

Let $J : X \times X \to \mathbb{R}_+$ be a symmetric measurable function. Let us define quadratic form $(\mathcal{E}_J, \mathcal{D})$ as follows

$$\mathcal{E}_J(f,f) = \frac{1}{2} \int\limits_{X \times X} (f(x) - f(y))^2 \, J(x,y) dm(x) dm(y). \tag{3.21}$$

We study the Markov generator $(L^J, \mathcal{D})$ defined by the kernel $J(x,dy) = J(x,y)dm(y)$. The operator $(L^J, \mathcal{D})$ we define either weakly, i.e. via representation

$$\mathcal{E}_J(f,f) = (L^J f, f), \tag{3.22}$$

or pointwise

$$L^J f(x) = \int\limits_X (f(x) - f(y)) \, J(x,dy). \tag{3.23}$$

In order to justify (3.21), (3.22) and (3.23) we assume that

(J.4) There exists an isotropic function $\mathcal{J}(x,y) = j(x \curlywedge y)$ with $j(B)$ satisfying $(J.1)$, $(J.2)$ and $(J.3)$, and such that

$$J(x,y) \asymp \mathcal{J}(x,y) \ \text{ uniformly in } \ x,y \in X.$$

Theorem 3.1 *Under condition $(J.4)$ the quadratic form $(\mathcal{E}_J, \mathcal{D})$ defined by equation (3.21) is closable and its closure is a regular Dirichlet form having $\mathcal{D}$ as a core. In particular, there exists a non-negative definite self-adjoint operator L^J such that $\mathcal{D} \in \mathrm{Dom}_{L^J}$ and for f in $\mathcal{D}$ equations (3.22) and (3.23) hold.*

Remark 3.2 *The conditions $(J.1)$, $(J.2)$ and $(J.3)$ imply that the isotropic Markov semigroup $(e^{-tL^{\mathcal{J}}})_{t \geq 0}$ acts in C_0 and admits a heat kernel $p^{\mathcal{J}}(t,x,y)$. Whether $(J.4)$ by itself implies that the L^2-Markov semigroup $(e^{-tL^J})_{t \geq 0}$ admits a heat kernel is an open problem at the present writing.*

Next theorem gives a partial answer to the question above. It is an *ultrametric* version of the celebrated Aronson '67 theorem for uniformly elliptic operators in $\mathbb{R}^d$.

Theorem 3.3 *Assume that $(J.4)$ holds. Assume that uniformly in $x \in X$ the volume function $r \to V(x,r)$ defined by the hierarchical Laplacian $L^{\mathcal{J}}$ is both doubling and reverse doubling. Then the L^2-Markov semigroup $(e^{-tL^J})_{t \geq 0}$*

acts in $C_0(X)$ and admits a Hölder continuous heat kernel $p^J(t,x,y)$. Moreover, uniformly in $x, y \in X$ and $t > 0$,

$$p^J(t,x,y) \asymp \min\left\{\frac{1}{V(x,t)}, \frac{t}{V(x,d_*(x,y))d_*(x,y)}\right\}, \tag{3.24}$$

where d_* is the intrinsic ultrametric defined by the operator $L^{\mathcal{J}}$.

Proof of Theorem 3.3 is based on recent papers

- Z.-Q. Chen, T. Kumagai and J. Wang, Stability of heat-kernel estimates for symmetric jump processes on metric spaces, arXiv 14 Apr 2016.
- A. Bendikov, A. Grigor'yan and E. Hu, Heat kernels and non-local Dirichlet forms on ultrametric spaces. Preprint 2017, 55 pp.

4 The p-adic setting

Recall that any translation invariant hierarchical Laplacian on $(\mathbb{Q}_p, d, m)$ can be represented in the form $\Phi(\mathfrak{D})$, where $\mathfrak{D} = \mathfrak{D}^1$ and $\Phi(\tau)$ is an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ can be written in terms of the Fourier transform as

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(|\xi|_p)\widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p, \ f \in \mathcal{D}.$$

As $\Phi(\mathfrak{D})$ is a hierarchical Laplacian,

$$\Phi(\mathfrak{D})f(x) = \int\limits_{\mathbb{Q}_p} (f(x) - f(y))\mathcal{J}_\Phi(x,y)dm(y), \quad f \in \mathcal{D},$$

Theorem 4.1 *Let $p_\Phi(t,x,y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assume that both Φ and Φ^{-1} are doubling, then*

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{|x-y|_p}\Phi\left(\frac{1}{|x-y|_p}\right)\right\}, \tag{4.25}$$

and

$$\mathcal{J}_\Phi(x,y) \asymp \frac{1}{|x-y|_p}\Phi\left(\frac{1}{|x-y|_p}\right) \tag{4.26}$$

uniformly in $t > 0$ and $x, y \in \mathbb{Q}_p$.

Theorem 4.2 *Let Φ be as above. Let $J(x,y)$ be a symmetric measurable function such that*

$$J(x,y) \asymp \frac{1}{|x-y|_p}\Phi\left(\frac{1}{|x-y|_p}\right) \text{ uniformly in } x,y \in X.$$

Then the operator $(L^J, \mathcal{D})$ extends to minus C_0-generator of symmetric Markov semigroup. This semigroup admits a Hölder continuous heat kernel $p^J(t,x,y)$ and the following estimates hold

$$p^J(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{|x-y|_p}\Phi\left(\frac{1}{|x-y|_p}\right)\right\} \tag{4.27}$$

uniformly in $t > 0$ and $x, y \in \mathbb{Q}_p$.

Symmetric infinitely divisible distributions A probability measure μ is said to be infinitely divisible if there exists a weakly continuous convolution semigroup of probability measures $(\mu_t)_{t\geq 0}$ such that $\mu = \mu_1$.

In terms of the Fourier transform $(\mu_t)_{t\geq 0}$ is characterised as follows

$$\widehat{\mu_t}(\theta) = e^{-t\psi(\theta)},\ \theta \in \mathbb{Q}_p,$$

where $\psi : \mathbb{Q}_p \to \mathbb{C}$ is a negative definite function such that $\psi(0) = 0$.

Assume that the measure μ is symmetric, then $\psi(\theta)$ is real non-negative and, by the the Lévy-Khinčin formula,

$$\psi(\theta) = \int\limits_{\mathbb{Q}_p\setminus\{0\}} (1 - \cos 2\pi\theta y)dJ(y).$$

Here J is a symmetric Radon measure on the set $\mathbb{Q}_p\setminus\{0\}$ (the Lévy measure).

Clearly, the Markov semigroup $P_t f = f * \mu_t$ is symmetric, acts in C_0, and its minus generator L can be written in the form

$$Lf(x) = \int (f(x) - f(x+y))dJ(y).$$

Let us assume that the measure J is absolutely continuous w.r.t. the Haar measure and that, for certain Φ as in Theorem 4.1, its density $J(y)$ satisfies

$$J(y) \asymp \frac{1}{|y|_p}\Phi\left(\frac{1}{|y|_p}\right) \text{ uniformly in } y.$$

Then, by Theorem 4.2, each measure μ_t has a Lipschitz continuous density $\mu_t(y)$ w.r.t. the Haar measure, and

$$\mu_t(y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{|y|_p}\Phi\left(\frac{1}{|y|_p}\right)\right\} \text{ uniformly in } t, y.$$

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